

# Stochastic Motion of an Open Bosonic String

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## Abstract

We show that the classical stochastic motion of an open bosonic string leads to the same results as the standard first quantization of this system. For this, the diffusion constant governing the process has to be proportional to  $\alpha'$ , the Regge slope parameter, which is the only constant, along with the velocity of light, needed to describe the motion of a string.

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## I. INTRODUCTION

String theory [1] provides an example of the not always straight path towards new ideas in physics. From its inception in the framework of dual-resonance models [2] to its current position as a candidate for a unified theory of all interactions, including gravity, it has gone through several modifications and increasing sophistication in its mathematical formulation. As a consequence of these changes, the original motivation for strings was abandoned and what we have now is a description of physics at the Planck scale [3], bringing with it all uncertainties on the validity of the basic quantum mechanical ideas. For sure we lack a clear physical picture of phenomena at such small distances, despite recent efforts by several authors [4] who introduce a radical view of physics at this scale, expanding the traditional quantum mechanical view. Some authors have gone as far as to speculate that at the Planck scale, quantum and thermal fluctuations cannot be distinguished [5].

Motivated by the above considerations we decided to look at a classical string subjected to stochastic motion, very much in the spirit of Nelson's stochastic approach to the motion of a classical newtonian particle [6]. This formulation achieves the derivation of a Schrödinger equation for a classical non-relativistic particle moving in an external potential by a stochastic version of Newton's second law of motion. There have been several criticisms to Nelson's alternative to quantum mechanics [7,8] and we do not want to address neither the criticisms nor the arguments purporting to defend it [9]. However, we think that if stochastic fluctuations, whose nature are never specified by Nelson and his followers, have any chance of manifesting themselves, then the natural place for them to occur would be at the very small scales like the Planck scale [5]. Since the objects that are candidates to describe physics at this scale are strings, we will in the following describe the stochastic motion of an open bosonic string, as the simplest such an object.

This paper is organized as follows. In the next section we briefly review Nelson's stochastic mechanics and very succinctly describe the stochastic variational principle, developed by Guerra and his coworkers [10], which will be useful in section III, where our stochastic formulation of an open bosonic string is presented. Among the results obtained, we can mention the derivation of a wave-functional equation for the string, the existence of a critical dimension ( $D = 26$ ), obtained through the requirement of Lorentz invariance (as we work in the light-cone gauge) and the two-point correlation function for the non-zero normal modes of the string. Our conclusions are in section IV.

## II. STOCHASTIC MECHANICS

### A. Nelson's approach

The starting point of Nelson's approach is to consider the stochastic motion of a point particle (for simplicity we treat a one dimensional motion) given by

$$dq(x, t) = v_+(q(t), t) dt + dw(t) , \quad (1)$$

where the first term on the right-hand side is deterministic and introduces a velocity field for forward propagation ( $dt > 0$ ) written as

$$v_+(x, t) = \frac{\nabla S_+(x, t)}{m} , \quad (2)$$

with  $m$  the particle mass and  $S_+$  a scalar function. The stochastic process is described by  $dw(t)$ , which satisfies the following averages

$$\langle dw(t) \rangle = 0 \quad (3)$$

$$\langle dw(t) dw(t) \rangle = 2\nu dt . \quad (4)$$

In (4)  $\nu$  is a diffusion constant to be specified later.

Given the non-differentiable nature of (1), Nelson then introduces the mean backward and forward transport derivatives

$$(D_+q)(x, t) = \lim_{\Delta t \rightarrow 0^+} \frac{\langle q(t + \Delta t) - q(t) \rangle}{\Delta t} = v_+(x, t) \quad (5)$$

$$(D_-q)(x, t) = \lim_{\Delta t \rightarrow 0^+} \frac{\langle q(t) - q(t - \Delta t) \rangle}{\Delta t} = v_-(x, t) , \quad (6)$$

which for a function  $F$  of  $x$  and  $t$  can be written, using (1) and (4) as

$$(D_\pm F)(x, t) = (\partial_t F)(x, t) + v_\pm(x, t)(\nabla F)(x, t) \pm \nu(\nabla^2 F)(x, t) . \quad (7)$$

(1) and (4) also imply the Fokker-Planck equation

$$\partial_t p(x, t; x_0, t_0) = -\nabla(v_+(x, t)p(x, t; x_0, t_0)) + \nu \nabla^2 p(x, t; x_0, t_0) , \quad (8)$$

where  $p$  is the transition probability, which, by definition, propagates the probability density of an ensemble of particles

$$\rho(x, t) = \int p(x, t; x_0, t_0) \rho(x_0, t_0) dx_0 . \quad (9)$$

With  $v_+$  and  $v_-$  we can also define

$$v = \frac{1}{2}(v_+ + v_-) = \frac{\nabla S}{m} \quad (10)$$

$$u = \frac{1}{2}(v_+ - v_-) = \nu \frac{\nabla \rho}{\rho} \quad (11)$$

and then obtain the continuity equation

$$\partial_t \rho(x, t) = -\nabla(\rho(x, t)v(x, t)) . \quad (12)$$

Nelson's formulation of the second law is [8]

$$\frac{1}{2}[(D_+D_- + D_-D_+)q](x, t) = -\frac{1}{m}(\nabla V)(x) . \quad (13)$$

From this equation of motion follows a Madelung type equation

$$\partial_t S + \frac{(\nabla S)^2}{2m} - 2m\nu^2[(\nabla R)^2 + \nabla^2 R] = -V(x) , \quad (14)$$

where  $R$  is related to the probability density  $\rho$  as follows

$$\exp(2R(x, t)) = \rho(x, t) . \quad (15)$$

The Madelung equation and the continuity equation will correspond respectively to the real and imaginary parts of a Schrödinger equation when writing the wave function in polar form

$$\psi(x, t) = \exp(R(x, t)) \exp\left(i \frac{S(x, t)}{\hbar}\right) \quad (16)$$

provided the diffusion constant  $\nu$  is identified with  $\frac{\hbar}{2m}$  [12,8].

## B. Stochastic Variational Principle

In order to avoid the ambiguity of defining the acceleration in Newton's second law, Guerra and collaborators [10] formulated stochastic mechanics with a variational principle. We refer the reader to ref. [10] for more details, since here we only need their lagrangian density which will lead to a Madelung equation.

Starting from

$$L(x, t) = \frac{1}{2}mv_+(x, t)v_-(x, t) - V(x) \quad (17)$$

and defining an average stochastic action

$$< A(t_0, t; \rho_0; v_+) > = \int_{t_0}^t \int L(x, t) \rho(x, t) dx dt \quad (18)$$

where  $\rho_0$  is the initial distribution, another lagrangian density depending only on  $v_+$  can be introduced

$$L_+(x, t) = \frac{1}{2}mv_+^2(x, t) + m\nu(\nabla v_+)(x, t) - V(x) . \quad (19)$$

It is possible to show that the action in equation (18) is the same as would be obtained replacing  $L$  by  $L_+$  (the extra terms in (19) vanish when taking the stochastic average).

Using a smooth field  $S_1$  and  $B(t_0, t; \rho_0, S_1; v_+)$  defined as

$$B(t_0, t; \rho_0, S_1; v_+) = A(t_0, t; \rho_0; v_+) - < S_1(q(t_1)) > = - \int J(x_0, t_0; t_1, S_1; v_+) \rho_0(x_0) dx_0 \quad (20)$$

we obtain

$$(D_+ J)(x, t) = L_+(x, t) \quad (21)$$

The variational principle based on  $\delta B = 0$  gives  $v(x, t) = (\nabla S)(x, t)/m$  (where  $S$  is  $J$  making  $B$  stationary) and allows the identification of (21) with the Madelung equation.

The continuity equation is the same as before and with the above Madelung equation, thus reproduce the Schrödinger equation (imaginary and real parts respectively). Furthermore, the variational approach leads naturally to a canonical stochastic formulation with  $\rho$

and  $S$  as canonical variables and, as shown by Guerra and Marra [11], to a redefinition of Poisson brackets in this context. We will make use of this formalism at the end of section III.

A final remark is important before we proceed to the stochastic description of the open bosonic string. In this section we summarized the stochastic approach to the non-relativistic point particle. In dealing with a string we should use a relativistic generalization of the Markov process underlying Nelson's stochastic formalism. This has been achieved in the literature [13] and in the following we will use this generalization.

### III. STOCHASTIC MOTION OF A STRING

We start from the Nambu-Goto action for a string

$$S = -\frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\pi d\sigma \sqrt{-\left(\frac{\partial x}{\partial \tau}\right)^2 \left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial x}{\partial \tau} \frac{\partial x}{\partial \sigma}\right)^2}, \quad (22)$$

which is proportional to the area of the two dimensional surface (embedded in a  $D$  dimensional space-time) swept by a string. The space-time points  $x^\mu(\sigma, \tau)$  on this surface are labelled by  $\sigma$  and  $\tau$ , two dimensionless parameters.  $\alpha'$  is the Regge slope which in string theory is related to the square of a fundamental length [3].

The above action is invariant under reparametrizations of the surface:  $\tilde{\sigma} \rightarrow \sigma = \sigma(\tilde{\sigma}, \tilde{\tau})$ ,  $\tilde{\tau} \rightarrow \tau = \tau(\tilde{\sigma}, \tilde{\tau})$ , which introduces a gauge freedom into the theory.. This leads to difficulties in the canonical formalism already at the classical level and one is forced to fix the gauge before proceeding. We choose to work in the light-cone gauge [14] in our classical approach. This choice of gauge will make Lorentz invariance not manifest and this will have to be addressed by the stochastic approach.

In the light-cone system one introduces coordinates as follows,

$$x^+ = \frac{x^0 + x^{D-1}}{\sqrt{2}} \quad (23)$$

$$x^- = \frac{x^0 - x^{D-1}}{\sqrt{2}} \quad (24)$$

and fixes the gauge as

$$x^+(\sigma, \tau) = p^+ \tau. \quad (25)$$

The transverse coordinates, the physical degrees of freedom, satisfy the following equation of motion,

$$\ddot{x}_i - x_i'' = 0 \quad (i = 1, \dots, D-2) \quad (26)$$

where  $\dot{x} = \frac{\partial x}{\partial \tau}$  and  $x' = \frac{\partial x}{\partial \sigma}$ .

We can now approach the system described by (26) from a stochastic point of view. We do this by expanding  $x_i(\sigma, \tau)$  in normal modes

$$x_i = \sum_{n=0}^{\infty} x_{ni} \cos n\sigma \quad (27)$$

(we are following the standard convention of defining  $\sigma$  between 0 and  $\pi$ ). From which follows,

$$\ddot{x}_{ni} + n^2 x_{ni} = 0 . \quad (28)$$

We now promote the normal modes to a stochastic process [15]  $q_n^i[x_m^i, \tau]$  with  $m = 0, \dots, \infty$  and  $i = 1, \dots, D-2$ , satisfying

$$dq_n^i[x_m^i, \tau] = v_{+n}^i d\tau + dw_n^i , \quad (29)$$

where  $dw_n^i$  obeys,

$$\langle dw_n^i(\tau) \rangle = 0 \quad (30)$$

$$\langle dw_n^i(\tau) dw_{n'}^j(\tau) \rangle = 2 \delta_{ij} \delta_{nn'} \nu_n d\tau . \quad (31)$$

Equation (29) illustrates the use of  $\tau$  as an evolution parameter.

Notice that we have not specified the diffusion constant  $\nu_n$  and have also allowed, for the sake of generality, a possible dependence on the normal mode index.

Following the steps outlined in section II we derive an equation of continuity

$$\partial_\tau \rho[x_m^i, \tau] = - \sum_n \nabla_n^i (\rho[x_m^i, \tau] v_n^i[x_m^i, \tau]) \quad (32)$$

where

$$v_n^i[x_m^i, \tau] = 4\alpha' \nabla_n^i S[x_m^i, \tau] \quad (33)$$

and for the zero mode

$$v_0^i[x_m^i, \tau] = 2\alpha' \nabla_0^i S[x_m^i, \tau] . \quad (34)$$

We can introduce, as before, the transport derivatives, which now read

$$D_\pm = \partial_\tau + \sum_n v_{\pm n}^i \nabla_n^i \pm \sum_n (\nabla_n^i)^2 \quad (35)$$

In the classical equation for the normal mode amplitudes

$$\ddot{q}_n^i + n^2 q_n^i = 0 \quad (36)$$

we use the definition of stochastic acceleration

$$\frac{1}{2}(D_+ D_- + D_- D_+) q_n^i = -n^2 q_n^i . \quad (37)$$

The Madelung equation that follows is

$$\begin{aligned} \partial_\tau S = & -\alpha'(\nabla_0^i S)^2 + \frac{\nu_0 \nu_n}{2\alpha'} [(\nabla_0^i R)^2 + (\nabla_0^i)^2 R] - \sum_{m \neq 0} 2\alpha'(\nabla_m^i S)^2 + \\ & + \frac{\nu_n \nu_n}{2\alpha'} \sum_{m \neq 0} [(\nabla_m^i R)^2 + (\nabla_m^i)^2 R] - \frac{1}{4\alpha'} \sum_m \frac{m^2 (x_m^i)^2}{2} = 0 . \end{aligned} \quad (38)$$

It satisfies, together with (32), the wave functional equation for a string (using  $\psi = \exp(R) \exp(\imath S)$ )

$$\imath \partial_\tau \psi = \left[ -\alpha'(\nabla_0^i)^2 + \sum_{n \neq 0} (-2\alpha'(\nabla_n^i)^2 + \frac{n^2 (x_n^i)^2}{8\alpha'}) \right] \psi , \quad (39)$$

provided the diffusion constants are

$$\nu_n = 2\alpha' \quad n \neq 0 \quad (40)$$

$$\nu_0 = \alpha' . \quad (41)$$

The difference between  $\nu_n$  and  $\nu_0$  comes from the convention used for separating the zero mode.

Notice the fact that we started this analysis without knowing the diffusion constant governing the stochastic process (31). For consistency reasons it results that in the case of the stochastic motion of a string the diffusion constant is  $\alpha'$ , which accords with the point of view stressed by Veneziano that a stringy world has only two constants  $c$  and  $\lambda$ , which is related to  $\alpha'$  as  $\lambda^2 = 2\alpha'$  [3]. If we naively expected  $\nu$  to be related to the energy of the system, like in point particle mechanics,  $\nu$  would be different for each state of the string. This encourages speculations on the indistinguishability of quantum and classical fluctuations at the Planck scale.

It follows from (38) that we obtain the standard spectrum of an open bosonic string, given by an infinite set of harmonic oscillators, as expected.

### A. Lorentz Invariance

We must now address the question of Lorentz invariance. In order to do so we make use of the correspondence between stochastic Poisson brackets and quantum commutators [11]. To achieve this correspondence it is important that  $\rho$  and  $S$ , as defined above, be canonical variables, in which case for a pair of dynamical variables  $A$  and  $B$ , functionals of  $\rho$  and  $S$

$$\{A, B\}_s = \int \left( \frac{\delta A}{\delta \rho} \frac{\delta B}{\delta S} - \frac{\delta A}{\delta S} \frac{\delta B}{\delta \rho} \right) dx . \quad (42)$$

Guerra e Marra [11] established the following correspondence

$$\{A, B\}_s = \imath \langle [A_q, B_q] \rangle , \quad (43)$$

where  $A_q$  and  $B_q$  are quantum operators and  $A$  and  $B$  are classical variables. The average value on the right hand side of (43) is defined as

$$< [A_q, B_q] > = \int \psi^* [A_q, B_q] \psi dx . \quad (44)$$

This result can be easily extended to our case ( $dx = \prod_{i,m} dq_m^i$ ) and allows us to examine the issue of Lorentz invariance by computing the Poisson brackets for the generators of the Lorentz group now formulated in stochastic language. With

$$M^{\mu\nu} = \int d\sigma (x^\mu P^\nu - x^\nu P^\mu) \quad (45)$$

the critical element of the algebra is

$$\{M^{i-}, M^{j-}\} = i < [M_q^{i-}, M_q^{j-}] > . \quad (46)$$

In order to close the algebra at a classical, but stochastic level, it is required that  $D = 26$ , in agreement with the well known results in string theory [1].

## B. Correlation Function

In order to evaluate the correctness of our results, we calculate next the two-point correlation function using classical stochastic methods. We restrict the calculation to the non-zero normal modes in order to avoid the well-known infrared divergence [1]. For the ground state we have

$$< x^i(\tau) x^i(\tau') > = \int x^i p(x^i, \tau, x'^i, \tau') x'^i \rho_0(x'^i) dx^i dx'^i , \quad (47)$$

which gives

$$< x^i(\tau) x^i(\tau') > = (D - 2) 2\alpha' \sum_{n \neq 0} \frac{1}{n} \frac{\exp(n\tau')}{\exp(n\tau)} \quad (48)$$

This is the same result obtained from the standard first quantized string, as can be seen in ref.[1], if we continue  $\tau$  to the Euclidean domain.

## IV. CONCLUSION

We have shown in this paper that the classical stochastic motion of an open bosonic string leads to a wave functional equation that correctly describes the first quantized string and that for the consistency of our results requires the string to live in a 26 dimensional world. The agreement of our two-point correlation function calculated by classical stochastic methods, with the one obtained by the standard treatment, reinforces the soundness of our formulation. Moreover, our results exhibit the naturalness of the string constant in playing the role of the diffusion constant at such small scales.

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